

Novel symmetries in Weyl-invariant gravity with massive gauge field

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Abstract The background field method is used to linearize the Weyl invariant scalar-tensor gravity, coupled with a Stückelberg field. For a generic background metric, this action is found to be *not* invariant, under both diffeomorphism and generalized Weyl symmetry, the latter being a combination of gauge and Weyl transformations. Interestingly, the quadratic Lagrangian, emerging from a background of Minkowski metric, respects both the transformations, *independently*. Becchi-Rouet-Stora-Tyutin (BRST) symmetry of scalar-tensor gravity coupled with a Stückelberg-like massive gauge particle, possessing diffeomorphism and generalized Weyl symmetry, reveals that in both the cases, negative norm states with unphysical degrees of freedom *do* exist. We then show that, by combining diffeomorphism and generalized Weyl symmetries, all the ghost states decouple, thereby removing the unphysical redundancies of the theory. During this process, the scalar field does not represent any dynamic mode, yet modifies the *usual* harmonic gauge condition through non-minimal coupling with gravity.

1 Introduction

Field theoretic models with scale-invariance have been found useful in the context of various cosmological problems [1, 2]. Scalar-tensor gravity (STG) is one such theory, which possesses local scale-invariance and has been widely used in explaining the inflationary properties of the universe [3–5]. It has recently been shown that, the conserved current corresponding to the Weyl symmetry of STG, vanishes identically [6]. Thus, although the

STG descends from a more general theory of the standard model coupled to gravity [2], at the level of the corresponding Lagrangian, the pure Weyl symmetry has no dynamical role and can be eliminated by making suitable choice of field variables [7, 8]. The *physicality* of the relevant scalar field can be retrieved by considering the *geodesic incompleteness* of such theories [9]. We, however, limit to the STG Lagrangian with no physical mode associated with the Weyl symmetry. In a recent paper, we have shown that there exists a massive scale invariant theory, with non-vanishing generalized scale current, when a Stückelberg field is suitably coupled with STG. It is imbued with a generalized Weyl symmetry: a combination of gauge and local scale symmetry [10]. Remarkably, the STG-Stückelberg theory, with gravity non-minimally coupled to the scalar field in STG, is a consistent theory only when the Weyl and gauge transformations are inter-related, making STG a part of the bigger theory. This aspect may find physical implication in scale-invariant cosmology. As is well known, gravitational theories are notorious to quantize, lacking gauge-fixing with respect to the diffeomorphism redundancy. Therefore, the classical analysis first needs to be corroborated with a proper gauge-fixing, for establishing the consistency of the still-extended STG-Stückelberg theory.

Quantization of gravity has been a challenge due to the highly non-linear nature of gravity. In this process, several important methods e.g., tetrad formalism, background field method etc., have been developed [11, 12]. Despite of its limitations due to the inherent simplicity, the background field approach has widely been exploited in various gravitational models [13–15]. The fact that, the background fields obey classical equations of motion is utilized to attain the lowest order, tree-level quantum action to be quadratic in nature, yield-

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ing equations of motion, linear in field variables. For the cases of pure gravity [12], as well as gravity, minimally coupled to a dilaton scalar field variables [13], such equations were obtained by identifying proper gauge-fixing against the inherent diffeomorphism redundancy of such systems under generic infinitesimal coordinate transformations. Recently, the *canonical* quantization of massive conformal gravity has been performed in the weak quantum field limit, revealing that the quantum massive gravity is unitary and renormalizable, but having ghost states [14]. Furthermore, using the same background field expansion, a generic Weyl invariant gravitational model with dilatations, in the presence of a gauge field, has also been quantized through suitable definition of renormalization group flows [15]. The latter case, including the STG suitably coupled with Abelian gauge field, is of importance in understanding light-matter interaction during conformal phases of the universe (e.g., inflation) and in higher dimensional gravity models with compactification (e.g., Kaluza-Klein gravity and several string theory models). However, even in these background field models, exact gauge-fixing of the conformal dilation gravity, coupled to gauge field, has not been *derived* so far. Further, consistent separation of the unphysical states, by introduction of proper ghost terms, remains undone till now.

In the present work, we consistently gauge-fix the STG-Stückelberg theory [10], depicting conformal gravity coupled to massive gauge particle, with the Stückelberg scalar field identified with the scaling scalar of STG, in the background field method. The equations of motion, obtained from a quadratic tree-level ‘quantum’ Lagrangian, are linear in field variables. Suitable choice of gauge-fixing yields corresponding dynamics with positive semi-definite norm. We adopt the celebrated Becchi-Rouet-Stora-Tyutin (BRST) method, where unitarity and gauge-invariance co-exist [16–19], to separate-out negative-norm states, leaving the rest of the dynamic fields ‘physical’. This is the first instance of BRST gauge fixing of such a theory to the best of our knowledge. Though this method allows for construction of quantum brackets at the tree (Lagrangian) level, it is generally accepted [11] that this further ensures consistent quantization up to all orders of perturbation. However, presently we leave the proof of the same for later works, and limit ourselves to the tree-level dynamics, with physical states identified. To this end, the corresponding Faddeev-Popov [20] ghosts are identified, with invariance under *two* independent supersymmetric transformations, BRST and anti-BRST [21–24], which are nilpotent and anticommuting. This enables us to identify the *effective* physical degrees of freedom of the theory, represented by graviton and massive gauge par-

ticle, with no dynamics for the scalar field. It is observed that this can be consistent only when Weyl and gauge symmetries are inter-related with diffeomorphism in a Minkowski background.

The paper is organized as follows. In Sec. 2, we apply the weak field method, with a generic metric as a background and linearize the generalized Weyl invariant STG-Stückelberg Lagrangian. Further, we show that quadratic Lagrangian is neither invariant under the usual diffeomorphism nor under the generalized Weyl transformation. Remarkably, on taking the background metric to be Minkowskian, the quadratic Lagrangian regains both the symmetries, which is depicted in Sec. 3. Subsequently, Sec. 4 is devoted to the discussion of (anti-)BRST symmetries of the quadratic Lagrangian, emerging from the STG-Stückelberg theory. We show that, for the removal of negative norm states, one has to combine both diffeomorphism and generalized Weyl transformations, in an appropriate manner. The harmonic gauge-fixing condition gets modified to a specific form of the well-known de Donder gauge due to the non-minimal coupling of gravity with the scalar field. We conclude after summarizing the obtained result and point out future directions of work.

2 Background field expansion

High non-linearity of Einstein’s gravity, prevents dynamical quantization of the same. A full non-perturbative approach has given rise to the $SL(2, \mathbb{R})$ gravity [11], described through the tetrad formalism [25]. However, high mathematical complexity has prevented the same from yielding practical solutions. Perturbative approaches are relatively simpler, which are based on the physical assumption of smallness of quantum fluctuations against classical curvature. This is very much plausible in a macroscopic universe, such as the present one. The background field approach [13] is applied to expand the complete metric in powers of quantum fluctuations $h_{\mu\nu}$ [12], with the corresponding dynamics governed by their second order contributions, as the first order terms vanish since the classical background fields are always taken to be on-shell [13]. As the corresponding Lagrangian is second order, the equations of motion are linear in $h_{\mu\nu}$, leading to linearized gravity [12].

Quantization of gravity requires, in general, the removal of the inherent redundancy arising from the symmetry under diffeomorphism transformation,

$$\begin{aligned} \delta_\xi A_{\mu\nu\dots}^{\alpha\beta\dots} &= \xi^\rho \partial_\rho A_{\mu\nu\dots}^{\alpha\beta\dots} + \xi^\alpha \partial_\rho A_{\mu\nu\dots}^{\rho\beta\dots} + \xi^\beta \partial_\rho A_{\mu\nu\dots}^{\alpha\rho\dots} + \dots \\ &\quad - \xi^\rho \partial_\mu A_{\rho\nu\dots}^{\alpha\beta\dots} - \xi^\rho \partial_\nu A_{\mu\rho\dots}^{\alpha\beta\dots}. \end{aligned} \quad (1)$$

This makes gravity a gauge theory, with a Fock space containing negative norm (ghost) states, needing proper

gauge-fixing. It is a difficult task in this highly non-perturbative theory [11]. In linearized gravity, the obvious approximation of the $SL(2, \mathbb{R})$ diffeomorphism transformation does reduce to one, that respects the symmetry of the second order Lagrangian; but only in the case of pure gravity [12]. In general, the presence of other fields coupled to gravity, that also transform under diffeomorphism, do not allow the same to be the symmetry of the full theory. This makes the theory non-covariant over general coordinate transformation. Therefore, very restricted boundary conditions are required [14, 15] to quantize a linear gravitational model with generic matter coupling, while preserving covariance over diffeomorphism.

As a primary need for quantization, by separating physical degrees of freedom from the unphysical ones, We proceed to gauge-fix the combined STG-Stückelberg theory [10],

$$\mathcal{L}_{\text{STSt}} = \sqrt{-\mathcal{G}} \left[\frac{1}{12\kappa} \mathcal{R} \Phi^2 + \frac{1}{2\kappa} \partial_\mu \Phi \partial_\nu \Phi \mathcal{G}^{\mu\nu} - \frac{1}{4} \mathcal{F}_{\mu\alpha} \mathcal{F}_{\nu\beta} \mathcal{G}^{\mu\nu} \mathcal{G}^{\alpha\beta} + \frac{m^2}{2} \mathcal{G}^{\mu\nu} \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi \right], \quad (2)$$

depicting a generalized Weyl-invariant massive gauge field in presence of curvature. Here, $\kappa = (16\pi G)$ is a constant of dimension inverse of square of the mass, with G being Newtonian gravitational constant, and R is the Ricci scalar. Such a model can be useful in explaining the behavior of a conformal, scale invariant universe having short-range gauge interactions. Here, the scalar field Φ couples non-minimally to gravity, defined by the metric $\mathcal{G}_{\mu\nu}$, in the first term of the above Lagrangian. As explained above, quantization of the above theory is much more difficult than Einstein-Hilbert gravity, given the nature of different couplings, including that of the Abelian gauge field A_μ represented by the corresponding field tensor $\mathcal{F}_{\mu\nu}$. Further, it should be noted that $\mathcal{D}_\mu = \partial_\mu - A_\mu$, a non- $U(1)$ covariant derivative corresponding to the gauge connection [10]. We will opt for quadratic form of this action, following the background field expansions:

$$\begin{aligned} \mathcal{G}_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} = g_{\mu\rho} (\delta_\nu^\rho + h_\nu^\rho), \\ \mathcal{G}^{\mu\nu} &= g^{\mu\nu} - h^{\mu\nu} + h^\mu_\rho h^{\rho\nu}, \\ h^{\mu\nu} &= h^{\nu\mu}, \quad \mathcal{G}_{\mu\nu} \mathcal{G}^{\nu\rho} = \delta_\mu^\rho = g_{\mu\nu} g^{\nu\rho}; \\ A_\mu &= a_\mu + b_\mu; \quad \Phi = \varphi + \phi. \end{aligned} \quad (3)$$

The first terms in the above are classical fields, obeying classical equations of motion (EOM) [10],

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= -m^2 \varphi^2 a^\nu + m^2 \varphi \partial^\nu \varphi, \\ \square \varphi &= \frac{1}{1 + \kappa m^2} \left[\frac{1}{6} R + \kappa m^2 (\nabla_\mu a_\nu + a_\mu a_\nu) g^{\mu\nu} \right] \varphi, \\ \mathcal{T}_{\mu\nu} &= \frac{1}{\kappa} \left[\frac{1}{12} G_{\mu\nu} \varphi^2 + \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right. \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{12} (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) \varphi^2 \Big] - \frac{1}{2} F_\mu^\alpha F_{\nu\alpha} \\ &+ \frac{m^2}{2} \varphi^2 \bar{a}_\mu \bar{a}_\nu - \frac{1}{4} g_{\mu\nu} \left(-\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \right. \\ &\left. + m^2 \varphi^2 \bar{a}_\alpha \bar{a}^\alpha \right) = 0; \end{aligned} \quad (4)$$

with,

$$\begin{aligned} \square \varphi &= \frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi], \quad \bar{a}_\mu = a_\mu - \partial_\mu \log \varphi, \\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \end{aligned} \quad (5)$$

The second terms in Eqs. (3) are quantum fluctuations, which are orders of magnitude suppressed. Here, we will follow the treatment by 't Hooft and Veltman [13], carried out for gravity minimally coupled with a scalar field, to obtain the most general second order quantum contribution to the tree-level Lagrangian of Eq. (2).

The most non-trivial term of the action, in view of background field expansion, is the first one, with gravity non-minimally coupled to the scalar field, which will be treated in detail. The background (classical) metric $g_{\mu\nu}$ raises and lowers indices and non-trivially contributes to the background covariant derivative D_μ . The gravitational Jacobian expands as [13],

$$\begin{aligned} \sqrt{-\mathcal{G}} &= \sqrt{-g} \left(1 + \frac{1}{2} h - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} + \frac{1}{8} h^2 \right); \\ h &= h^\mu_\mu, \end{aligned} \quad (6)$$

where we have used the second expression from the first equation of Eqs. (3). Then the first term in Eq. (2) can be expanded, in the powers of the ‘quantum’ metric $h_{\mu\nu}$ and scalar field ϕ , in the form,

$$\mathcal{L}_g := \frac{1}{12\kappa} \sqrt{-\mathcal{G}} \mathcal{R} \Phi^2 \equiv \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{O}(3), \quad (7)$$

wherein \mathcal{L}_0 has no ‘quantum dynamics’. \mathcal{L}_1 is first order in $h_{\mu\nu}$ and ϕ , and its contribution is exactly canceled by that coming from the first order contribution from the rest of the Lagrangian, following the classical equations of motion [13]. Hence, the lowest order quantum dynamics arises from \mathcal{L}_2 , which is quadratic in quantum fields, and can be decomposed into the three parts $\mathcal{L}_2 = \mathcal{L}_a + \mathcal{L}_b + \mathcal{L}_c$, with the expressions,

$$\begin{aligned} \mathcal{L}_a &= \frac{\sqrt{-g}}{12\kappa} {}^0 R \phi^2, \\ \mathcal{L}_b &= \frac{\sqrt{-g}}{6\kappa} \left(h^{\mu\nu}{}_{;\mu\nu} - \square h + h^{\mu\nu} {}^0 R_{\mu\nu} - \frac{1}{2} h {}^0 R \right) \varphi \phi, \\ \mathcal{L}_c &= \frac{\sqrt{-g}}{12\kappa} \left[\frac{1}{8} (h^2 - 2h^{\mu\nu} h_{\mu\nu}) {}^0 R - h^{\alpha\mu} h_{\mu\nu} {}^0 R^\nu_\alpha \right. \\ &\quad + \frac{1}{2} h h^{\mu\nu} {}^0 R_{\mu\nu} + \frac{1}{4} h^{\mu\nu} \square h_{\mu\nu} - \frac{1}{4} h \square h + \frac{1}{2} h h^{\mu\nu}{}_{;\mu\nu} \\ &\quad \left. + \frac{1}{2} h^{\mu\nu;\alpha} h_{\mu\alpha;\nu} \right] \varphi^2. \end{aligned} \quad (8)$$

The last expression is obtained through integration by parts and removing total derivatives, with commutator for covariant derivatives being taken into account. Here, the semi-colon indicates covariant derivative with respect to the background metric. Moreover, ${}^0R_{\mu\nu}$ and 0R represent the Ricci tensor and scalar with respect to the background metric, respectively.

Expanding the rest of the Lagrangian in Eq. (2), we get,

$$\begin{aligned}\mathcal{L}_{\text{STSt}}^{(2)} \equiv & \mathcal{L}_2 + \frac{\sqrt{-g}}{2\kappa} \left[\frac{1}{8} (h^2 - 2h^{\mu\nu} h_{\mu\nu}) \varphi_{,\alpha} \varphi^{,\alpha} \right. \\ & + \frac{h}{2} (2\varphi_{,\mu} \phi^{,\mu} - h^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}) + \varphi_{,\mu} h^\mu{}_\alpha h^{\alpha\nu} \varphi_{,\nu} \\ & - 2h^{\mu\nu} \varphi_{,\mu} \phi_{,\nu} + \phi_{,\mu} \phi^{,\mu} \Big] - \frac{\sqrt{-g}}{4} \left[\frac{1}{8} (h^2 \right. \\ & - 2h^{\mu\nu} h_{\mu\nu}) A^{\alpha\beta} A_{\alpha\beta} + h \left(A_{\mu\nu} B^{\mu\nu} \right. \\ & - h^{\mu\nu} A_\mu{}^\alpha A_{\nu\alpha} \Big) + 2A_{\mu\alpha} h^\mu{}_\beta h^{\beta\nu} A^\alpha{}_\nu \\ & + h^{\mu\nu} h^{\alpha\beta} A_{\mu\alpha} A_{\nu\beta} + B_{\mu\nu} B^{\mu\nu} - 2h^{\mu\nu} A_\mu{}^\alpha B_{\nu\alpha} \Big] \\ & + \frac{\sqrt{-g}}{2} m^2 \left[\frac{1}{8} (h^2 - 2h^{\mu\nu} h_{\mu\nu}) d_\alpha \varphi d^\alpha \varphi + d_\mu \phi d^\mu \phi \right. \\ & - \frac{1}{2} h h^{\mu\nu} d_\mu \varphi d_\nu \varphi + d_\mu \varphi h^\mu{}_\alpha h^{\alpha\nu} d_\nu \varphi - 2b_\mu \varphi d^\mu \phi \\ & - b_\mu \phi d^\mu \varphi + (h g^{\mu\nu} - 2h^{\mu\nu}) (d_\mu \varphi d_\nu \phi - b_\mu \varphi d_\nu \varphi) \\ & \left. + b_\mu b^\mu \varphi^2 \right] + \mathcal{O}(h^3) \\ & = \mathcal{L}_2 + \mathcal{L}_\phi + \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{int}}.\end{aligned}\quad (9)$$

This action is quadratic in quantum fields and represents their dynamics. Here, $A_{\mu\nu}$ and $B_{\mu\nu}$ correspond to the background and quantum gauge field strengths, respectively and $d_\mu := \partial_\mu - a_\mu$.

The present theory, being coupled to gravity, is expected to be invariant under infinitesimal general coordinate transformation of the form,

$$x^\mu \rightarrow x^\mu + \xi^\mu, \quad (10)$$

with ξ^μ being small. The corresponding variation of the fields is diffeomorphism, under which, background fields do not change, whereas the quantum fields do [12–14] as [12, 13],

$$\begin{aligned}h_{\mu\nu} & \rightarrow h_{\mu\nu} + \xi_{(\mu;\nu)} + \xi^\alpha{}_{;\mu} h_{\nu\alpha} + \xi^\alpha h_{\mu\nu;\alpha}, \\ h^{\mu\nu} & \rightarrow h^{\mu\nu} - \xi^{(\mu;\nu)} - \xi^{\alpha;(\mu} h^\nu{}_{\alpha)} - \xi^\alpha h^{\mu\nu}{}_{;\alpha}, \\ h & \rightarrow h + 2\xi^\mu{}_{;\mu} + 2h^{\mu\nu} \xi_{\mu;\nu} + \xi^\mu h_{;\mu}, \\ b_\mu & \rightarrow b_\mu + \xi^\rho \partial_\rho A_\mu + A_\rho \partial_\mu \xi^\rho, \\ \phi & \rightarrow \phi + \xi^\rho \partial_\rho \Phi.\end{aligned}\quad (11)$$

Here ξ^μ is identified as the diffeomorphism parameter. As observed earlier [13], if smallness of the quantum fields are *not* taken into account, *i.e.* the Lagrangian being considered up to all orders of expansion, the transformations in Eqs. (11) represent symmetry of the full

Lagrangian, not only that of the quadratic part. Since the present goal is concerned with the second order contribution, yielding linear EOMs, one needs to reduce these transformations suitably up to certain orders. The obvious choice is to restrict the R.H.S. of Eqs. (11) to zero order in quantum fields, as ξ_μ is small [12]. However, unlike pure gravity, the presence of other interacting dynamic fields in the present case forbids such a reduced symmetry of the second order Lagrangian in Eq. (9).

The same is true for the independent generalized Weyl transformations [10], which can be attributed completely to the quantum fields [14] as,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - 2\theta \mathcal{G}_{\mu\nu}, \quad b_\mu \rightarrow b_\mu + \partial_\mu \theta, \quad \phi \rightarrow \phi + \theta \Phi, \quad (12)$$

which is a symmetry of the full Lagrangian in Eq. (2). For the second order Lagrangian, this is no more a symmetry, even when approximated to zero order in quantum fields. This imposes strict restrictions on the background fields, in order to retain both diffeomorphism and generalized Weyl symmetries, in reduced forms, in the Lagrangian. In the following, we obtain such a second order Lagrangian with specific background (classical) fields.

3 Second order theory with constant background

Following the case of pure gravity [12] and that of conformal scalar-tensor gravity [14], wherein a reduced diffeomorphism is obtainable in the second order theory for a constant (Minkowski) background metric, for the present case, we consider all three background fields to be constant to yield,

$$\begin{aligned}\mathcal{G}_{\mu\nu} & = \eta_{\mu\nu} + h_{\mu\nu}, \quad \Phi = \varphi + \phi, \quad a_\mu = b_\mu, \\ \sqrt{-g} & = 1 + \frac{1}{2}h - \frac{1}{4}h^{\mu\nu} h_{\mu\nu} + \frac{1}{8}h^2 + \dots,\end{aligned}\quad (13)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $a_\mu = 0$. The latter, more restrictive choice owes to the fact that a constant non-zero, non-dynamic vector field destroys the isotropy of the background space, when coupled to the dynamic fields. Further, the generalized Weyl symmetry is broken in presence of a vector that does not transform. These approximations simplify the second order Lagrangian, further as,

$$\begin{aligned}\mathcal{L}_{\text{STSt}}^{(2)\text{red}} & = \frac{1}{12\kappa} \left[\frac{1}{4} h_{\mu\nu} \square h^{\mu\nu} + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\alpha h^{\mu\alpha} \partial^\beta h_{\mu\beta} \right. \\ & - \frac{1}{4} h \square h \Big] \varphi^2 + \frac{1}{6\kappa} \left[\partial_\mu \partial_\nu h^{\mu\nu} - \square h \right] \varphi \phi \\ & + \frac{1}{2\kappa} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 \left[\partial_\mu \phi \partial^\mu \phi \right. \\ & \left. - 2b_\mu \varphi \partial^\mu \phi + b_\mu b^\mu \varphi^2 \right].\end{aligned}\quad (14)$$

Under the reduced diffeomorphism transformations,

$$\delta_\xi h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta_\xi b_\mu = 0, \quad \delta_\xi \phi = 0, \quad (15)$$

with quantum fields b_μ and ϕ unchanged, the simplified second order Lagrangian is invariant, as it changes by a total derivative term,

$$\begin{aligned} \delta_\xi \mathcal{L}_{\text{STSt}}^{(2)\text{red}} = & \partial_\mu \left[\frac{1}{24\kappa} \left\{ (\partial_\nu \xi_\alpha) \partial^\mu h^{\nu\alpha} - \partial^\mu (\partial_\nu \xi_\alpha) h^{\nu\alpha} \right. \right. \\ & \left. \left. - (\partial_\alpha \xi^\alpha) \partial^\mu h + \partial^\mu (\partial_\alpha \xi^\alpha) h \right. \right. \\ & \left. \left. + 2(\partial_\alpha \xi^\alpha) \partial_\nu h^{\mu\nu} \right\} \varphi^2 \right]. \end{aligned} \quad (16)$$

The generalized Weyl transformation for the quantum fields reduces to,

$$\delta_\theta h_{\mu\nu} = -2\theta \eta_{\mu\nu}, \quad \delta_\theta b_\mu = \partial_\mu \theta, \quad \delta_\theta \phi = \theta \varphi, \quad (17)$$

which also leaves the Lagrangian in Eq. (14) invariant modulo a total derivative,

$$\begin{aligned} \delta_\theta \mathcal{L}_{\text{STSt}}^{(2)\text{red}} = & \partial_\mu \left[\frac{1}{24\kappa} (h \partial^\mu \theta - \partial^\mu h \theta - 4\theta \partial_\nu h^{\mu\nu}) \varphi^2 \right. \\ & \left. + \frac{1}{\kappa} \varphi \phi \partial^\mu \theta \right]. \end{aligned} \quad (18)$$

As both diffeomorphism and generalized Weyl transformations are independent, they can be combined as,

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\eta_{\mu\nu} \theta, \\ \delta b_\mu &= \partial_\mu \theta, \quad \delta \phi = \theta \varphi, \end{aligned} \quad (19)$$

with $\delta = (\delta_\xi + \delta_\theta)$, leaving the corresponding Lagrangian invariant modulo a total derivative term,

$$\begin{aligned} \delta \mathcal{L}_{\text{STSt}}^{(2)\text{red}} = & \partial_\mu \left[\frac{1}{24\kappa} \left\{ (\partial_\nu \xi_\alpha) \partial^\mu h^{\nu\alpha} - \partial^\mu (\partial_\nu \xi_\alpha) h^{\nu\alpha} \right. \right. \\ & \left. \left. - (\partial_\alpha \xi^\alpha) \partial^\mu h + \partial^\mu (\partial_\alpha \xi^\alpha) h + 2(\partial_\alpha \xi^\alpha) \partial_\nu h^{\mu\nu} \right\} \varphi^2 \right. \\ & \left. + \frac{1}{24\kappa} (h \partial^\mu \theta - \partial^\mu h \theta - 4\theta \partial_\nu h^{\mu\nu}) \varphi^2 \right. \\ & \left. + \frac{1}{\kappa} \varphi \phi \partial^\mu \theta \right]. \end{aligned} \quad (20)$$

As the Lagrangian in (14) is invariant, independently under approximated diffeomorphism and generalized Weyl transformations, *both* of which represent constraints, the physical degrees of freedom of this system is less than the number of field components. Direct quantization of such a system will lead to a Fock space with negative norm (energy) states, corresponding to unphysical degrees of freedom. This calls for gauge-fixing each of the two symmetries, so that the equations of motion corresponding to *each* field correspond only to positive energy states. To this end, the proper gauge-fixing conditions are to be obtained from the respective Euler-Lagrange equations for fields $h_{\mu\nu}$, b_μ and ϕ , which are,

$$\begin{aligned} \square h^{\mu\nu} + \eta^{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - \eta^{\mu\nu} \square h + \partial^\mu \partial^\nu h + \partial_\alpha \partial^\mu h^{\alpha\nu} \\ + \partial_\alpha \partial^\nu h^{\alpha\mu} &= \frac{4}{\varphi} \left(\eta^{\mu\nu} \square \phi - \partial^\mu \partial^\nu \phi \right), \\ \frac{1}{6\kappa} (\partial_\mu \partial_\nu h^{\mu\nu} - \square h) \varphi &= \left(\frac{1}{\kappa} + m^2 \right) \square \phi - m^2 \varphi \partial_\mu b^\mu, \\ \partial_\mu F^{\mu\nu} &= m^2 (\varphi \partial^\nu \phi - \varphi^2 b^\nu). \end{aligned} \quad (21)$$

In the gravity sector, the *extended* harmonic gauge choice [13],

$$\partial_\mu h^{\mu\nu} - \frac{1}{2} \partial^\nu h - \frac{2}{\varphi} \partial^\nu \phi = 0, \quad (22)$$

leads to the EOM for $h_{\mu\nu}$ as,

$$\square \bar{H}^{\mu\nu} = 0; \quad \bar{H}^{\mu\nu} := h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h - \frac{2}{\varphi} \eta^{\mu\nu} \phi. \quad (23)$$

Thus, the ‘gravitational’ mode is represented by a linear combination of quantum metric and scalar fields, which is expected from the non-minimal coupling between the two [7, 8, 13]. This physically makes sense, as the scalar mode correspond to both scaling and Stückelberg mechanism [10]. Hence, it should not represent any physical degree of freedom. This assertion will be clearer in the following and also in the next section, when (anti-) BRST symmetry will be obtained.

In the vector gauge sector, the independent covariant gauge choice of,

$$\partial_\mu b^\mu + m^2 \varphi \phi = 0, \quad (24)$$

much like the Stückelberg case [26], leads to the EOM for b_μ as,

$$(\square + m^2 \varphi^2) b^\nu = 0. \quad (25)$$

This particular gauge choice is considered due to the fact that, the gauge field couples to the scalar one, as in Eq. 2, just like the Stückelberg theory, with the coupling terms re-arranged. This further justifies acquired mass, $m_b = m\varphi$,

$$(26)$$

of the gauge field. Further, on using the gauge-fixing conditions in Eqs. (22) and (24), and quantum metric EOM of Eq. (23) thereby substituting $h_{\mu\nu}$, h and b_μ , the EOM for ϕ reduces to,

$$(\square + m^2 \varphi^2) \phi = 0, \quad (27)$$

with a ‘mass term’,

$$m_\phi = m\varphi \equiv m_b. \quad (28)$$

The last result is of importance, as it ensures that Eq. (27) can be obtained from Eq. (25) subjected to the gauge-fixing condition of Eq. (24). However, the reverse is *not* true. This further shows that ϕ does not represent an independent dynamical mode, as asserted earlier. The two gauge-fixing conditions relate the scalar field independently to the tensor and vector fields. Also, *both* these gauge-fixing conditions are required to obtain Eq. (27), and it can be written purely in terms of other two fields, which is *not* the case for Eqs. (23) and (25). Another way to validate this point is that the Lagrangian in Eq. (2) was obtained through a generalized Weyl scaling of a Proca action in presence of gravity [10], with the scalar component Φ exactly being the scaling field

[6] and also the Stückelberg-type field. In both regards, it is expected that the scalar component will not yield any dynamics.

Therefore, we have obtained a massless gravitational and massive vector field at the tree-level, subjected to the gauge-fixing conditions in Eqs. (22) and (24). This is in accordance to the earlier observations [7, 8, 13], with modification coming due to the additional gauge coupling [10]. As symmetric $h_{\mu\nu}$ has 10 independent components and the b_μ has 4; 4 modified harmonic constraints Eq. (22), 4 coming from the generic coordinate transformation Eq. (10) [12] and 1 vector gauge constraint Eq. (24) leave the net physical degrees of freedom of the system to be $(10 + 4) - (4 + 4 + 1) = 5$, with 2 for graviton and 3 for massive gauge particle, which is expected physically. This aspect will be further elucidated, while introducing the ghost fields for the theory, in the next section. As the exact gauge-fixing conditions have been achieved, by obtaining ‘physical’ equations of motion (23) and (25) with positive spectra, the (anti-)BRST symmetries of the present system can be carried out, which is demonstrated in the next section.

4 (Anti-)BRST transformations

Our goal is to identify the physical degrees of freedom of the present theory, which has redundancies arising from both diffeomorphism and generalized Weyl transformations. To this end, we opt for the BRST approach, wherein auxiliary gauge-fixing conditions are chosen at the expense of introducing Faddeev-Popov ghost and anti-ghost fields, representing negative norm states arising from the gauge redundancies of the original theory [20]. Effectively, this increase in variables re-locates the over-counting of degrees of freedom from the gauge to the (anti-)ghost sector, as gauge transformation is replaced by the corresponding (anti-)BRST transformations [16–19]. The latter is nilpotent and as it is done off-shell, one is left with only physical degrees of freedom in the gauge sector, defined in a Fock space with positive semi-definite norm.

In the present case, the diffeomorphism gauge sector contains only one vector ghost term in the Lagrangian, since it is confined only to $h_{\mu\nu}$, with ϕ being non-dynamic. Similarly, the generalized Weyl redundancy will lead to *one* scalar ghost term, that affects all the three fields, including b_μ and ϕ . This is because the gauge-fixing term for $h_{\mu\nu}$ remains invariant under the corresponding nilpotent (anti-)BRST transformations. Finally, the combined diffeomorphism and generalized Weyl gauge sector is dealt with, achieving the physical states of the full theory, through necessary inclusion of both vector and scalar (anti-)ghost fields.

4.1 For diffeomorphism symmetry

Here, we consider the path-integral (anti-)BRST symmetry of the Lagrangian with respect to diffeomorphism. This is exclusive to the gravitational sector, through introduction of corresponding vector (anti-)ghost fields, made possible by its independence from the generalized Weyl symmetry in the present theory. For this purpose, we gauge-fix only the gravitational sector of the complete Lagrangian in Eq. (14), in the modified harmonic gauge of Eq. (22) and introduce corresponding vector (anti-)ghost fields $(\bar{c}_\mu)c_\mu$, with ghost numbers $(-1)1$, to yield,

$$\begin{aligned} \mathcal{L}_{\text{GF1}}^{(2)} = & \frac{1}{12\kappa} \left[\frac{1}{4} h_{\mu\nu} \square h^{\mu\nu} + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\alpha h^{\mu\alpha} \partial^\beta h_{\mu\beta} \right. \\ & - \frac{1}{4} h \square h \Big] \varphi^2 + \frac{1}{6\kappa} \left[\partial_\mu \partial_\nu h^{\mu\nu} - \square h \right] \varphi \phi + \frac{1}{2\kappa} \partial_\mu \phi \partial^\mu \phi \\ & - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 \left[\partial_\mu \phi \partial^\mu \phi - 2 b_\mu \varphi \partial^\mu \phi + b_\mu b^\mu \varphi^2 \right] \\ & - \frac{1}{24\kappa} \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h - 2\varphi^{-1} \partial^\mu \phi \right)^2 \\ & + \frac{i}{12\kappa} \partial^\mu \bar{c}^\nu \partial_\mu c_\nu. \end{aligned} \quad (29)$$

Here, and afterwards, we explicitly use the Feynman-’t Hooft gauge. The above Lagrangian can be linearized by adding the Nakanishi-Lautrup type auxiliary vector field B_μ as,

$$\begin{aligned} \mathcal{L}_{\text{GF1}}^{(2)} \equiv & \frac{1}{12\kappa} \left[\frac{1}{4} h_{\mu\nu} \square h^{\mu\nu} + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\alpha h^{\mu\alpha} \partial^\beta h_{\mu\beta} \right. \\ & - \frac{1}{4} h \square h \Big] \varphi^2 + \frac{1}{6\kappa} \left[\partial_\mu \partial_\nu h^{\mu\nu} - \square h \right] \varphi \phi \\ & + \frac{1}{2\kappa} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 \left[\partial_\mu \phi \partial^\mu \phi \right. \\ & - 2 b_\mu \varphi \partial^\mu \phi + b_\mu b^\mu \varphi^2 \Big] + \frac{1}{24\kappa} B_\mu B^\mu \\ & + \frac{1}{12\kappa} B_\mu \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h - 2\varphi^{-1} \partial^\mu \phi \right) \\ & + \frac{i}{12\kappa} \partial^\mu \bar{c}^\nu \partial_\mu c_\nu. \end{aligned} \quad (30)$$

with $B_\mu = -\left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h - 2\varphi^{-1} \partial^\mu \phi\right)$ on-shell. The action corresponding to the above Lagrangian is invariant under the following off-shell nilpotent (anti-)BRST transformations,

$$\begin{aligned} s_b^{(1)} h_{\mu\nu} &= \partial_\mu c_\nu + \partial_\nu c_\mu, & s_b^{(1)} c_\mu &= 0, & s_b^{(1)} \bar{c}_\mu &= i B_\mu, \\ s_b^{(1)} B_\mu &= 0 & s_b^{(1)} b_\mu &= 0, & s_b^{(1)} \phi &= 0, \\ s_{ab}^{(1)} h_{\mu\nu} &= \partial_\mu \bar{c}_\nu + \partial_\nu \bar{c}_\mu, & s_{ab}^{(1)} \bar{c}_\mu &= 0, & s_{ab}^{(1)} c_\mu &= -i B_\mu, \\ s_{ab}^{(1)} B_\mu &= 0 & s_{ab}^{(1)} b_\mu &= 0, & s_{ab}^{(1)} \phi &= 0, \end{aligned} \quad (31)$$

as the Lagrangian transforms by total derivative terms, respectively, as,

$$s_b^{(1)} \mathcal{L}_{\text{GF1}}^{(2)} = \partial_\mu \left[\frac{1}{24\kappa} \left\{ (\partial_\nu c^\alpha) \partial^\mu h^{\nu\alpha} - \partial^\mu (\partial_\nu c_\alpha) h^{\nu\alpha} \right\} \right]$$

$$\begin{aligned}
& -(\partial_\alpha c^\alpha)\partial^\mu h + \partial^\mu(\partial_\alpha c^\alpha)h + 2(\partial_\alpha c^\alpha)\partial_\nu h^{\mu\nu}\Big\}\varphi^2 \\
& + \frac{1}{12\kappa}B^\nu\partial^\mu c_\nu\Big], \\
s_{ab}^{(1)}\mathcal{L}_{\text{GF1}}^{(2)} = & \partial_\mu\Big[\frac{1}{24\kappa}\Big\{(\partial_\nu\bar{c}^\alpha)\partial^\mu h^{\nu\alpha} - \partial^\mu(\partial_\nu\bar{c}_\alpha)h^{\nu\alpha} \\
& - (\partial_\alpha\bar{c}^\alpha)\partial^\mu h + \partial^\mu(\partial_\alpha\bar{c}^\alpha)h + 2(\partial_\alpha\bar{c}^\alpha)\partial_\nu h^{\mu\nu}\Big\}\varphi^2 \\
& - \frac{1}{12\kappa}B^\nu\partial^\mu\bar{c}_\nu\Big]. \tag{32}
\end{aligned}$$

Therefore, the gauge redundancy corresponding to diffeomorphism is removed by the inclusion of the gauge-fixing term, whereas the corresponding symmetry is now compensated by the (anti-)BRST symmetry of the (anti-)ghost fields. The physical credibility of the above procedure is reflected by the fact that, in the original Lagrangian of Eq. (14), the symmetric tensor field $h_{\mu\nu}$ had 10 degrees of freedom, unlike the physical graviton, that has only 2. To identify the correct number, one has to go to the level of equations of motion (on-shell) to figure out 8 constraint equations, 4 from harmonic gauge and the other four from the remaining choice of the corresponding parameter ξ_μ in coordinate transformation, as discussed in the previous section. On the other hand, introduction of vector (anti-)ghost fields, with four *negative* degrees of freedom each, corresponds to the same physics at the level of the Lagrangian itself (off-shell). The auxiliary field is always on-shell and does not have any dynamics. This enables $h_{\mu\nu}$ in $\mathcal{L}_{\text{GF1}}^{(2)}$ to represent exact physical degrees of freedom and thus, with only positive energy states in the Fock space, with the negative energy ones attributed to the (anti-)ghost fields.

It is to be noted that, only the gravitational sector is considered above, mainly to demonstrate the physical aspects of the same. Eq. (30) represents a two-component tensor field, with semi-positive norm, coupled to the fields b_μ , with gauge redundancy, and ϕ . Thus, only the prior can represent tree-level quantum dynamics. To achieve the full spectrum of physical states for $\mathcal{L}_{\text{STst}}^{(2)\text{red}}$, we need to fix the generalized Weyl redundancy corresponding to the other two fields, which is done later.

4.2 For generalized Weyl symmetry

In order to gauge-fix the generalized Weyl symmetry, which includes tensor, vector and scalar sectors of $\mathcal{L}_{\text{STst}}^{(2)\text{red}}$, unlike diffeomorphism before, we extend $\mathcal{L}_{\text{STst}}^{(2)\text{red}}$ to,

$$\begin{aligned}
\mathcal{L}_{\text{GF2}}^{(2)} = & \frac{1}{12\kappa}\Big[\frac{1}{4}h_{\mu\nu}\Box h^{\mu\nu} + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu} + \frac{1}{2}\partial_\alpha h^{\mu\alpha}\partial^\beta h_{\mu\beta} \\
& - \frac{1}{4}h\Box h\Big]\varphi^2 + \frac{1}{6\kappa}\Big[\partial_\mu\partial_\nu h^{\mu\nu} - \Box h\Big]\varphi\phi + \frac{1}{2\kappa}\partial_\mu\phi\partial^\mu\phi \\
& - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{2}m^2\Big[\partial_\mu\phi\partial^\mu\phi - 2b_\mu\varphi\partial^\mu\phi + b_\mu b^\mu\varphi^2\Big]
\end{aligned}$$

$$\begin{aligned}
& + \frac{B^2}{2} + B\Big(\partial\cdot b + m^2\varphi^2\phi\Big)^2 \\
& + i\partial^\mu\bar{c}\partial_\mu c + im^2\varphi^2\bar{c}c. \tag{33}
\end{aligned}$$

with scalar (anti-)ghost (\bar{c}) c and Nakanishi-Lautrup auxiliary scalar B , with $B = -(\partial\cdot b + m^2\varphi^2\phi)$ on-shell [27]. The above Lagrangian, under the (anti-)BRST transformations,

$$\begin{aligned}
s_b^{(2)}h_{\mu\nu} &= -2c\eta_{\mu\nu}, & s_b^{(2)}b_\mu &= \partial_\mu c, & s_b^{(2)}\phi &= \varphi c, \\
s_b^{(2)}c &= 0, & s_b^{(2)}\bar{c} &= iB, & s_b^{(2)}B &= 0, \\
s_{ab}^{(2)}h_{\mu\nu} &= -2\bar{c}\eta_{\mu\nu}, & s_{ab}^{(2)}b_\mu &= \partial_\mu\bar{c}, & s_{ab}^{(2)}\phi &= \varphi\bar{c}, \\
s_{ab}^{(2)}\bar{c} &= 0, & s_{ab}^{(2)}c &= -iB, & s_{ab}^{(2)}B &= 0. \tag{34}
\end{aligned}$$

changes by the total derivative terms,

$$\begin{aligned}
s_b^{(2)}\mathcal{L}_{\text{GF2}}^{(2)} &= \partial_\mu\Big[\frac{1}{24\kappa}(h\partial^\mu c - \partial^\mu hc - 4c\partial_\nu h^{\mu\nu})\varphi^2 \\
& + \frac{1}{\kappa}\varphi\phi\partial_\mu c + B\partial^\mu c\Big], \\
s_{ab}^{(2)}\mathcal{L}_{\text{GF2}}^{(2)} &= \partial_\mu\Big[\frac{1}{24\kappa}(h\partial^\mu\bar{c} - \partial^\mu h\bar{c} - 4\bar{c}\partial_\nu h^{\mu\nu})\varphi^2 \\
& + \frac{1}{\kappa}\varphi\phi\partial_\mu\bar{c} - B\partial^\mu\bar{c}\Big], \tag{35}
\end{aligned}$$

respectively, leaving the corresponding action invariant.

As in the case of diffeomorphism before, only partial physical spectrum has been achieved, as is evident from the corresponding equations of motion, with only Eq. (25) being reproduced, yielding positive energy states only for the vector field b_μ . The fact that the gauge-fixing condition in Eq. (24) relates b_μ with the scalar field ϕ makes the 2 negative degrees of freedom of the (anti-)ghost scalars compensating *together* against the redundant combination of 4 positive degrees of freedom of b_μ and 1 of ϕ . This leaves three physical degrees of freedom for b_μ , as ϕ does not represent a dynamical mode, analogous to the Stückelberg theory [28], having a positive spectrum and thereby being the only physical field, corresponding to the particular symmetry, coupled to $h_{\mu\nu}$ that contains un-physical states presently. Therefore, It is clear that though diffeomorphism and generalized Weyl gauge-fixings are mutually independent, all the dynamical components (field) of the present theory cannot be made physical, at the same time, by any one of them. A direct evidence of the same is the fact that, Eq. (27) for ϕ is *not* obtained in any of these ‘independent’ procedure, and thus the scalar degree of freedom remains ill-defined in both of them. This is because, in the second order theory, ϕ is the only field, out of the three, that directly couples to the other two. Therefore, the positive-norm Fock space for the full theory can only be achieved with both diffeomorphism and generalized Weyl gauge-fixings applied, which is achieved below.

4.3 For combined diffeomorphism and generalized Weyl symmetry

As discussed above, in order to obtain the complete physical spectrum of the theory, both the gauge-fixing conditions in Eqs. (22) and (24) are to be applied. Including the corresponding auxiliary Nakanishi-Lautrup and (anti-)ghost fields, of both vector and scalar types, the full tree-level quantum Lagrangian can be written as,

$$\begin{aligned}\mathcal{L}_{\text{GF}}^{(2)} = & \frac{1}{12\kappa} \left[\frac{1}{4} h_{\mu\nu} \square h^{\mu\nu} + \frac{1}{2} h \partial_\mu \partial_\nu h^{\mu\nu} + \frac{1}{2} \partial_\alpha h^{\mu\alpha} \partial^\beta h_{\mu\beta} \right. \\ & - \frac{1}{4} h \square h \Big] \varphi^2 + \frac{1}{6\kappa} \left[\partial_\mu \partial_\nu h^{\mu\nu} - \square h \right] \varphi \phi + \frac{1}{2\kappa} \partial_\mu \phi \partial^\mu \phi \\ & - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} m^2 \left[\partial_\mu \phi \partial^\mu \phi - 2b_\mu \varphi \partial^\mu \phi + b_\mu b^\mu \varphi^2 \right] \\ & + \frac{1}{24\kappa} B_\mu B^\mu + \frac{1}{12\kappa} B_\mu \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h - 2\varphi^{-1} \partial^\mu \phi \right) \\ & + \frac{B^2}{2} + B \left(\partial \cdot b + m^2 \varphi^2 \phi \right)^2 + \frac{i}{12\kappa} \partial^\mu \bar{c}_\nu \partial_\mu c^\nu \\ & + i \partial^\mu \bar{c} \partial_\mu c + i m^2 \varphi^2 \bar{c} c. \end{aligned} \quad (36)$$

Thus, the unphysical ghost sector, imbibing negative norm states, is completely decoupled from the other dynamic fields, leaving them associated only with positive semi-definite energies, and thereby removing the gauge redundancies due to *both* diffeomorphism and generalized Weyl symmetries. The latter is now re-structured as the following BRST and anti-BRST transformations,

$$\begin{aligned} s_b h_{\mu\nu} &= \partial_\mu c_\nu + \partial_\nu c_\mu - 2\eta_{\mu\nu} c, & s_b b_\mu &= \partial_\mu c, \\ s_b \phi &= \varphi c, & s_b \bar{c}_\mu &= i B_\mu, & s_b \bar{c} &= i B, \\ s_b [c_\mu, c, B_\mu, B] &= 0, \\ s_{ab} h_{\mu\nu} &= \partial_\mu \bar{c}_\nu + \partial_\nu \bar{c}_\mu - 2\eta_{\mu\nu} \bar{c}, & s_{ab} b_\mu &= \partial_\mu c, \\ s_{ab} \phi &= \varphi c, & s_{ab} c_\mu &= -i B_\mu, & s_{ab} c &= -i B, \\ s_{ab} [\bar{c}_\mu, \bar{c}, B_\mu, B] &= 0. \end{aligned} \quad (37)$$

Under these transformations, the Lagrangian $\mathcal{L}_{\text{GF}}^{(2)}$ respectively transforms by the total derivatives,

$$\begin{aligned} s_b \mathcal{L}_{\text{GF}}^{(2)} &= \partial_\mu \left[\frac{1}{24\kappa} \left\{ (\partial_\nu c^\alpha) \partial^\mu h^{\nu\alpha} - \partial^\mu (\partial_\nu c_\alpha) h^{\nu\alpha} \right. \right. \\ & \quad \left. \left. - (\partial_\alpha c^\alpha) \partial^\mu h + \partial^\mu (\partial_\alpha c^\alpha) h + 2(\partial_\alpha c^\alpha) \partial_\nu h^{\mu\nu} \right\} \varphi^2 \right. \\ & \quad \left. + \frac{1}{\kappa} \varphi \phi \partial_\mu c + \frac{1}{24\kappa} (h \partial^\mu c - \partial^\mu h c - 4c \partial_\nu h^{\mu\nu}) \varphi^2 \right. \\ & \quad \left. + \frac{1}{12\kappa} B^\nu \partial^\mu c_\nu + B \partial^\mu c \right], \\ s_{ab} \mathcal{L}_{\text{GF}}^{(2)} &= \partial_\mu \left[\frac{1}{24\kappa} \left\{ (\partial_\nu \bar{c}^\alpha) \partial^\mu h^{\nu\alpha} - \partial^\mu (\partial_\nu \bar{c}_\alpha) h^{\nu\alpha} \right. \right. \\ & \quad \left. \left. - (\partial_\alpha \bar{c}^\alpha) \partial^\mu h + \partial^\mu (\partial_\alpha \bar{c}^\alpha) h + 2(\partial_\alpha \bar{c}^\alpha) \partial_\nu h^{\mu\nu} \right\} \varphi^2 \right. \\ & \quad \left. + \frac{1}{\kappa} \varphi \phi \partial_\mu \bar{c} + \frac{1}{24\kappa} (h \partial^\mu \bar{c} - \partial^\mu h \bar{c} - 4\bar{c} \partial_\nu h^{\mu\nu}) \varphi^2 \right. \\ & \quad \left. - \frac{1}{12\kappa} B^\nu \partial^\mu \bar{c}_\nu - B \partial^\mu \bar{c} \right], \end{aligned} \quad (38)$$

leaving the action invariant. Both the relevant fields are now properly gauge-fixed, and satisfy ‘free’ equations (23), (25) and (27), which can lead to quantum effects, due to respective couplings.

Physically, it is crucial to evaluate the actual degrees of freedom of the complete Lagrangian $\mathcal{L}_{\text{GF}}^{(2)}$, in the (anti-)BRST formalism. We count 10 tensor ($h_{\mu\nu}$), 4 vector (b_μ) and 1 scalar (ϕ) component of dynamical fields, yielding 15 positive degrees of freedom (not considering the non-dynamic auxiliary fields). The negative degree of freedom count is 10 (4 vector and 1 scalar each for both ghost and anti-ghost kind). Therefore, the system has 5 physical degrees of freedom, 2 for $h_{\mu\nu}$ (graviton) and 3 for b_μ (massive gauge boson). This is the result we obtained in the last section, but at the level of equations of motion. This re-ensures the non-dynamical status of ϕ , governed by Eq. (27), as in the previous section.

Therefore, the physical domain of the STG-Stückelberg theory has been completely identified in the weak field approximation. This is done through the (anti-)BRST procedures, ensuring the same to be *off-shell*, thus enabling Euler-Lagrange equations derived out of the Lagrangian in Eq. (36) to yield proper tree-level propagators for both $\bar{H}_{\mu\nu}$ and b_μ . In the generalized graviton sector, the gauge-fixed EOM reads,

$$\begin{aligned} \frac{1}{2} \left[\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \partial^\nu \partial^\alpha + \eta^{\nu\beta} \partial^\mu \partial^\alpha \right. \\ \left. - \eta^{\mu\nu} \partial^\alpha \partial^\beta \right] \bar{H}_{\alpha\beta} = 0, \end{aligned} \quad (39)$$

which is exactly of the form for pure gravity under harmonic gauge [12], yielding the corresponding propagator,

$$\Delta^{\mu\nu, \alpha\beta} = -\frac{i}{2} \frac{\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}}{p^2 + i\epsilon}, \quad (40)$$

representing an effective massless mode [12, 13]. Similarly, the gauge propagator is similar to the Stückelberg propagator [28],

$$\Delta^{\mu\nu} = -i \frac{\eta^{\mu\nu}}{p^2 - m_b^2}. \quad (41)$$

The present gauge-fixing, of the quadratic Lagrangian, having both reduced diffeomorphism and generalized Weyl symmetry, is of the most general form and complements the results in [14, 15], especially those with canonical approach [14]. This is sufficient to quantize the theory at the tree-level, with *bare* propagators specified. However, the quantum corrections to a particular field, due to the interaction with another, can provide corrections to the bare structures, and even lead to anomalies. We leave such possibilities for future investigations.

5 Conclusions

In conclusion, the STG-Stückelberg theory with non-minimal scalar-gravity coupling, following background field method, led to inconsistencies for a not-trivial background metric; as *both* diffeomorphism, as well as generalized Weyl symmetry of the classical theory, are not respected. However, very interestingly, in Minkowskian background, this theory can be consistently gauge-fixed by retaining both the symmetries, where all the negative norm states (ghost states) are decoupled from the physical Fock space. Earlier, the quantization of massive conformal gravity has been carried out by showing that the theory is renormalizable but has ghost states [14]. Here, the ghost-free Fock space has been achieved, after relating all symmetries; namely diffeomorphism and generalized Weyl symmetry. Further, the graviton and massive gauge field represents the entire dynamics, and the scalar field remain non-dynamical in the present case.

Using the well-established BRST technique, the quadratic Lagrangian, emerging from the STG-Stückelberg theory, has been *independently* gauge-fixed, with respect to the diffeomorphism and generalized Weyl transformations. In the gravitational sector, corresponding to diffeomorphism only, an appropriately *extended* harmonic gauge condition, of the de Donder form, is required, yielding a pair of vector (anti-)ghost fields. This leaves-out the negative norm states with respect to the dynamical vector field b_μ in the theory. On the other hand, gauge-fixing only the extended Weyl gauge redundancy yields three physical degrees of freedom for the gauge field b_μ , as the scalar field ϕ is not dynamical. Thus, negative norm states exist in both the cases, and the complete physical sector of the STG-Stückelberg theory is not identified, preventing the corresponding quantization. The identification of the full physical Fock space of the quadratic STG-Stückelberg Lagrangian requires a vector and a scalar pair of (anti-)ghost terms, obtained by *combining* the corresponding transformations. All the unphysical degrees of freedom of fields $h_{\mu\nu}$ and b_μ have been removed, yielding the 5 physical degrees of freedom, 2 for graviton ($h_{\mu\nu}$) and 3 for the massive gauge boson (b_μ). It will be worthwhile to investigate effective theories emerging from the present one, with interesting possibilities in areas like conformal gravity and gauge theories, as well as in condensed-matter systems. Some of these questions will be investigated in future works.

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